TAKEAWAY TASKS NO 1

(1) Recall that an *inner product* on a real Hilbert space $\mathscr{H}$ is a bilinear function $\langle\cdot,\cdot\rangle : \mathscr{H}\times\mathscr{H}\to\mathbb{R}$ satisfying the following:
   - $\langle x,x\rangle \geq 0$ and the equality holds if and only if $x=0$,
   - $\langle x,y\rangle = \langle y,x\rangle$.

   Prove that the function $d\colon \mathscr{H}\times\mathscr{H}\to\mathbb{R}$ given by $d(x,y)=\sqrt{\langle x-y,x-y\rangle}$ is a metric on $\mathscr{H}$.

(2) Consider a set X_n of all sequences of length n whose values are either 0 or 1. The *Hamming distance* between two such sequences is the number of indices $1\leq i\leq n$ at which these sequences differ. Prove that Hamming distance is a metric on X_n. How do the balls in this metric look like?

(3) Let $\{a_0,a_1,\dots,a_n\}\subset\mathbb{R}^d$. Show that the following are equivalent:
   - $a_1-a_0,\dots,a_n-a_0$ are linearly independent,
   - if λ_i,μ_i satisfy $\sum_{i=0}^n\lambda_i a_i=\sum_{i=0}^n\mu_i a_i$ and $\sum_{i=0}^n\lambda_i=\sum_{i=0}^n\mu_i$, then $\lambda_i=\mu_i$ for $i=0,\dots,n$.

(4) Let S be a finite subset of simplices of a simplicial complex K. Prove that there exists a finite simplicial complex $L<K$ (that is, consisting of finitely many simplices) such that $S\subseteq L$.

(5) Denote by S^1 and D^2 the standard one-dimensional sphere and the two-dimensional disc, i.e. $S^1=\{(x,y)\in\mathbb{R}^2 | x^2+y^2=1\}$ and $D^2=\{(x,y)\in\mathbb{R}^2 | x^2+y^2\leq 1\}$. Let X be a topological space. Prove that the following conditions are equivalent:
   - every continuous function $f: S^1\to X$ is homotopic to a constant function,
   - every continuous function $f\colon S^1\to X$ extends to a continuous function $g: D^2\to X$ (i.e. the restriction of g to $S^1\subset D^2$ is f).