Nonvanishing of group cohomology of $\mathrm{SL}_n(\mathbb{Z})$

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joint work in progress with: Benjamin Brück (ETH Zürich/Universität Münster), Sam Hughes (University of Oxford), and Dawid Kielak (University of Oxford)

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- $H^1(\pi(\Sigma_g)) \cong \mathbb{Z}^{2g}$, $H^0(\pi(\Sigma_g)) \cong H^2(\pi(\Sigma_g)) \cong \mathbb{Z}$, $H^k(\pi(\Sigma_g)) \cong 0$ for $k \notin \{0, 1, 2\}$.

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Theorem (cf. Bekka – de la Harpe – Valette)

G (σ-compact and locally compact) has property (T) if and only if $H^1(G, \pi) = 0$ for every orthogonal representation π of G.

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A discrete group G has property $(\boldsymbol{T_n})$ if $H^k(G; \pi) = 0$ for every unitary G-representation π without nontrivial invariant vectors and $1 \leq k \leq n$.

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$$\pi(a) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \pi(M) = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 2 \\ -1 & 1 & 2 & 0 \end{bmatrix}$$

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Theorem (Brück – Hughes – Kielak – M.)

For $n = 3, 4$ there exist an orthogonal representation π_n of $\mathrm{SL}_n(\mathbb{Z})$ without nontrivial invariant vectors such that $H^{n-1}(\mathrm{SL}_n(\mathbb{Z}), \pi_n) \neq 0$.