Induction of spectral gaps for the cohomological Laplacians of $\mathrm{SL}_n(\mathbb{Z})$ and $\mathrm{SAut}(F_n)$

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joint work with Marek Kaluba (KIT)

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Outline

Motivation

Group cohomology and property (T)

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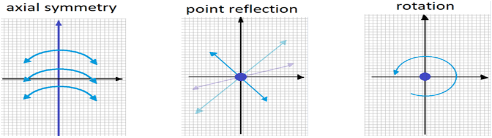

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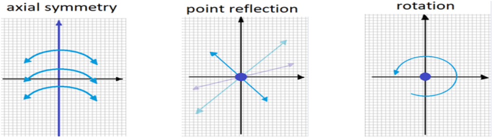

- Generalization: symmetries must have fixed points

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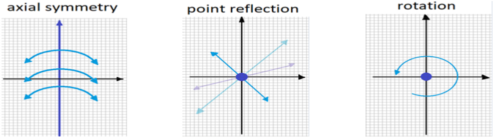

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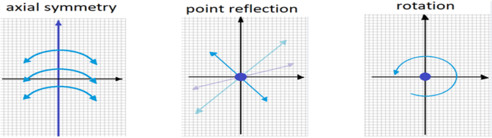

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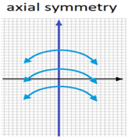

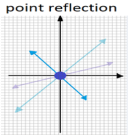

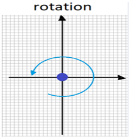

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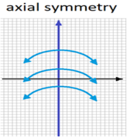

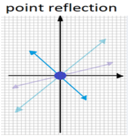

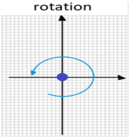

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Theorem (Shalom et. al.)

G has Kazhdan's property (T) if $H^1(G, \pi) = 0$ for every orthogonal representation $\pi : G \to \mathcal{B}(\mathcal{H})$.

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- (T) yields expanders: $G_n := G/N_n$, G has (T) (Margulis '80s)

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Theorem (Kaluba, Kielak, Nowak, Ozawa, 2019, 2021)

$\mathrm{Aut}(F_n)$ *has property (T) if* $n \geq 5$.

Group ring conditions and an application to $\mathsf{SL}_n(\mathbb{Z})$

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Theorem (Kaluba, M., 2023)

For $\mathrm{SL}_n(\mathbb{Z})$*, one has* $\Delta_1 - 0.217(n-2)I \geq 0$.

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- $\Delta \in \mathbb{R}G$, $\Delta_1 \in \mathbb{M}_{|S|\times|S|}(\mathbb{R}G)$

Induction – idea

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- $\Delta_1 = d_1^* d_1 + d_0 d_0^*$

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The differentials $\frac{\partial}{\partial s_j} : \mathbb{R}F_n \to \mathbb{R}G$, $F_n = \langle s_1, \ldots, s_n \rangle$ are defined by:

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Induction for Δ_1

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Proof.

- $\mathrm{Sq}_n^- = \sum_{1\leq i\neq j\leq n} d^{i,j}\left(d^{i,j}\right)^*$, $d^{i,j}$ – column vector with $1-s$ for s on indices i and j

□

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The matrices Sq_n^- *and* $\mathrm{Op}_n^+ + 2\,\mathrm{Op}_n^-$ *are positive.*

Proof.

- $\mathrm{Sq}_n^- = \sum_{1 \le i \ne j \le n} d^{i,j} \left(d^{i,j}\right)^*$, $d^{i,j}$ – column vector with $1 - s$ for s on indices i and j
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- the rest cancels out:

$$\left(\mathrm{Op}_n^+\right)_{\alpha_{ij},\beta_{kl}} = \left(\frac{\partial[\alpha_{ij},\beta_{kl}]}{\partial\alpha_{ij}}\right)^* \frac{\partial[\alpha_{ij},\beta_{kl}]}{\partial\beta_{kl}} + \left(\frac{\partial[\beta_{kl},\alpha_{ij}]}{\partial\alpha_{ij}}\right)^* \frac{\partial[\beta_{kl},\alpha_{ij}]}{\partial\beta_{kl}}$$
$$= -2(1-\alpha_{ij})(1-\beta_{kl})^*.$$

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- $\mathrm{Adj}_n \approx \sum_{\sigma\in\Sigma_n} \sigma\widetilde{\mathrm{Adj}_m}$ for $n \geq m$, provided Adj_m is Σ_m-invariant

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- invariance for Adj^+ follows from the equivariance of the Jacobian d_1

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Lemma

For any $\sigma \in \Sigma_m$, r, and s, we have $\frac{\partial r}{\partial(\sigma s)} = \sigma\left(\frac{\partial\left(\sigma^{-1} r\right)}{\partial s}\right)$.

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Corollary (Kaluba, M., 2023)

For $\mathrm{SL}_n(\mathbb{Z})$*, one has* $\Delta_1 - 0.217(n-2)I \geq 0$.

Final remarks

Spectral gaps for Δ and Δ_1

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- $d_1 d_0 = 0$
- $\Delta_1 - \lambda I \geq 0 \Rightarrow \Delta^2 - \lambda\Delta = d_0^* (\Delta_1 - \lambda I) d_0 \geq 0$

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- if we try to do the same for Adj part, we end up with negative λ ...
- ... on other hand, for $SL_4(\mathbb{Z})$, we have Adj $-0.009I = N_1^* N_1 + \dots N_l^* N_l$ for N_l supported on ball of radius 1

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- follow up problem: can we do something similar for Δ_1?

Thank you for your attention!