Fox derivatives, group cohomology, and higher property (T)

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GABY 2022

Introduction

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- We focus on finitely presented groups

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- Results

Relations between main concepts

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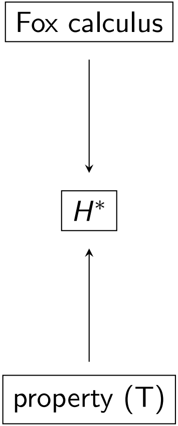
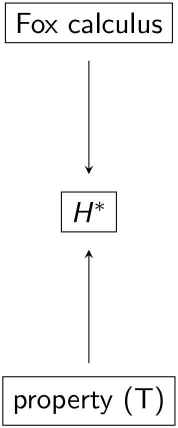
- Fox derivatives define group cohomology



Relations between main concepts

- Fox derivatives define group cohomology
- Kazhdan's property (T) is a cohomological property



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- Positivity in group rings (sums of squares)
- We work with matrices over $\mathbb{R}G$

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The differentials $\frac{\partial}{\partial s_j} : \mathbb{R}F_n \to \mathbb{R}G$, $F_n = \langle s_1, \ldots, s_n \rangle$ are defined by:

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The Fox derivatives are the elements $\frac{\partial r_i}{\partial s_j} \in \mathbb{R}G$.

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- $\mathbb{Z}^2 = \langle a, b | aba^{-1}b^{-1} \rangle$:

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Theorem (Lyndon, '50s)

The cohomology $H^*(G, V)$ is the cohomology of the following complex:

$$0 \to V \xrightarrow{d_0} V^n \xrightarrow{d_1} V^m \to \cdots .$$

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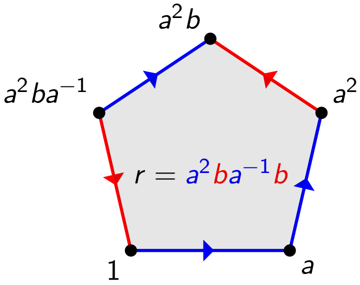

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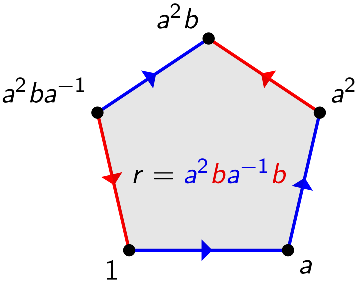

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Vanishing and reducibility of cohomology

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Theorem (Ozawa, 2014)

$G = \langle s_1, \ldots, s_n | \cdots \rangle$ has property (T) iff there exists $\lambda > 0$ such that $\Delta_0^2 - \lambda\Delta_0 = \mathrm{SOS}$ ($\Delta_0 = d_0^* d_0 = \sum_{i=1}^n (1 - s_i)^*(1 - s_i)$).

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Definition

The ith reduced cohomology is defined by $\overline{H}^i = \operatorname{Ker} d_i / \overline{\operatorname{Im} d_{i-1}}$.
We say that the ith cohomology is reduced if H^i coincides with $\overline{H}^i$.

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Proposition (Dymara-Januszkiewicz)

For any $i \geq 2$ there exists a group G_i with reduced H^i and $H^i(G, \rho_0) \neq 0$ for some unitary representation ρ_0.

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TFAE for G and $i \geq 1$:

- H^i *vanish and* H^{i+1} *are reduced.*
- $\Delta_i - \lambda I = \mathrm{SOS}$ *for some* $\lambda > 0$.

Results

(joint work with M. Kaluba and P. Nowak)

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- Convex optimization for $M = \Delta_1 - \lambda I$:

$$\begin{aligned} \text{maximize:} \quad & \lambda \\ \text{subject to:} \quad & M_{i,j}(g) = \langle \delta_{i,j} \otimes \delta_g, P \rangle, \\ & P \succeq 0. \end{aligned}$$

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Lemma

Suppose $\sum_{r \in R'} r^* r + d_0 d_0^* - \lambda I$ is an SOS. Then $\Delta_1 - \lambda I$ is an SOS as well.

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Proof.

Just add $\sum_{r \in R \setminus R'} r^* r$ to the expression $\sum_{r \in R'} r^* r + d_0 d_0^* - \lambda I$. □

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Corollary

The first cohomology of $\mathrm{SL}_3(\mathbb{Z})$ vanishes, and the second is reduced.

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- Expanders generalize to higher dimensions (Lubotzky)

A comment on expanders

- $G = (V, E)$
- Cheeger constant: $h(G) = \inf_{1 \leq \#A \leq \#V/2} \frac{\#E(A, V \setminus A)}{\#A}$
- Expander family: $|G_n| \to \infty$ s.t. $\liminf_{n \to \infty} \frac{h(G_n)}{\deg(G_n)} > 0$
- (T) yields expanders: $G_n := G/N_n$, G has (T)
- Expanders generalize to higher dimensions (Lubotzky)
- $SL_3(\mathbb{Z})$: spectral gap $\Rightarrow$ "CW-expanders"

Thank you for attention!