Spectral gaps for cohomological Laplacians of $SL_n(\mathbb{Z})$

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Theorem (Ozawa '14)

$H^1 \equiv 0 \iff \Delta^2 - \lambda\Delta \geqslant 0$ for some $\lambda > 0$,

where $\Delta = |S| - \sum_{s\in S} s \in \mathbb{R}G$.

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- How to show $H^n \equiv 0$,

  … not only for $n=1$?

$$\cdots \longrightarrow (\mathbb{R}G)^{k_{n-1}} \xrightarrow{D_{n-1}} (\mathbb{R}G)^{k_n} \xrightarrow{D_n} (\mathbb{R}G)^{k_{n+1}} \longrightarrow \cdots$$

$$
\begin{tikzcd}
\cdots \arrow[r] & (\mathbb{R}G)^{k_{n-1}} \arrow[r, "D_{n-1}"] \arrow[d, squiggly] & (\mathbb{R}G)^{k_n} \arrow[r, "D_n"] & (\mathbb{R}G)^{k_{n+1}} \arrow[r] \arrow[d, squiggly] & \cdots \\
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\cdots \arrow[r] & \mathcal{H}^{k_{n-1}} \arrow[r, "\pi(D_{n-1})"] & \mathcal{H}^{k_n} \arrow[r, "\pi(D_n)"] & \mathcal{H}^{k_{n+1}} \arrow[r] & \cdots
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\end{tikzcd}$$

Thm (Bader, Nowak '20)

$H^n \equiv 0$, provided $\Delta_n - \lambda I \geq 0$ for some $\lambda > 0$, where $\Delta_n = D_n^* D_n + D_{n-1} D_{n-1}^* \in M_{k_n}(\mathbb{R}G)$.

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Idea:

prove for low degrees and induce for higher using the elementary matrix presentation $G_n = \langle E_{ij}, \ldots \rangle$ and the decomposition $\Delta^2 = Sq + Adj + Op$.

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$$\Delta_1 - 0.32 I \geq 0 \text{ for } n = 3.$$

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- What about $SL_n(\mathbb{Z})$, $n \geqslant 4$?

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Thm (Kaluba, M. '23)

$Adj - 0.2(n-2)I \geqslant 0$. Hence $\Delta_1 - 0.2(n-2)I \geqslant 0$.

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Conclusions:

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Conclusions:

- alternative proof of (T) for $SL_n(\mathbb{Z})$

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Conclusions:

- alternative proof of (T) for $SL_n(\mathbb{Z})$
- induction method can be applied to $SAut(F_n)$ as well.