Fox derivatives, group cohomology, and higher property (T)

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Introduction

About the problem

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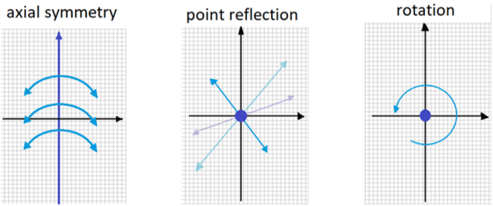

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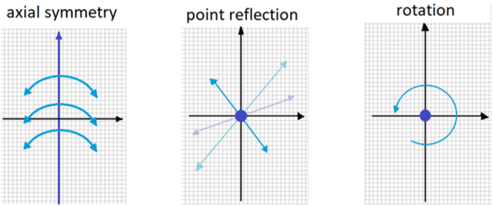

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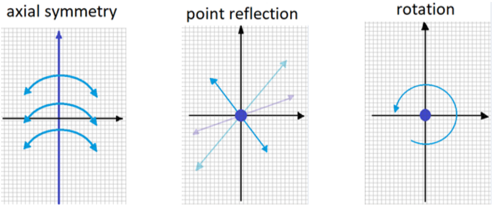

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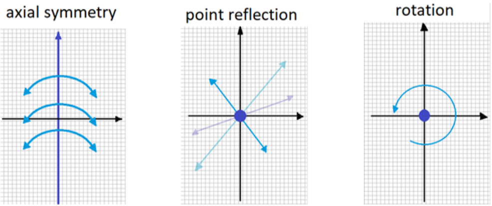

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- Property (T) and higher property (T) are related to fixed point properties
- Applications to expanders' constructions

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- Idea: interpretation in a group ring setting

Relations between main concepts

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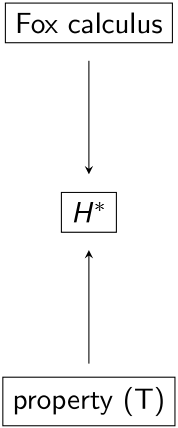
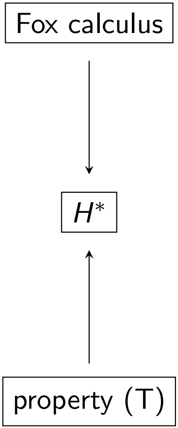
- Fox calculus computes group cohomology



Relations between main concepts

- Fox calculus computes group cohomology
- Kazhdan's property (T) is a cohomological property



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- We work with matrices over $\mathbb{R}G$

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- We decide SOS property with convex optimization

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Fox calculus

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The Fox derivatives are the elements $\frac{\partial r_i}{\partial s_j} \in \mathbb{R}G$.

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Theorem (Lyndon, '50s)

The cohomology $H^*(G, V)$ is the cohomology of the following complex:

$$0 \to V \xrightarrow{d_0} V^n \xrightarrow{d_1} V^m \to \cdots .$$

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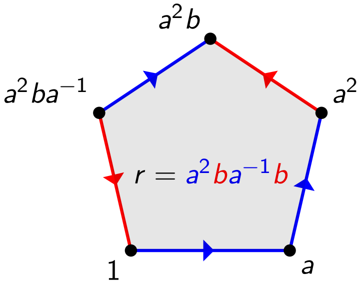

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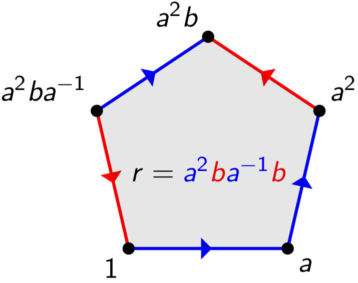

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Vanishing and reducibility of cohomology

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Theorem (Ozawa, 2014)

$G = \langle s_1, \ldots, s_n | \cdots \rangle$ has property (T) iff there exists $\lambda > 0$ such that $\Delta_0^2 - \lambda\Delta_0 = \mathrm{SOS}$ ($\Delta_0 = d_0^* d_0 = \sum_{i=1}^n (1 - s_i)^*(1 - s_i)$).

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The ith reduced cohomology is defined by $\overline{H}^i = \operatorname{Ker} d_i / \overline{\operatorname{Im} d_{i-1}}$.
We say that the ith cohomology is reduced if H^i coincides with $\overline{H}^i$.

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Proposition (Dymara-Januszkiewicz)

For any $i \geq 2$ there exists a group G_i with reduced H^i and $H^i(G, \rho_0) \neq 0$ for some unitary representation ρ_0.

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TFAE for G and $i \geq 1$:

- *H^i vanish and H^{i+1} are reduced.*
- *$\Delta_i - \lambda I = \text{SOS}$ for some $\lambda > 0$.*

Results

(joint work with M. Kaluba and P. Nowak)

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- Convex optimization for $M = \Delta_1 - \lambda I$:

$$\begin{aligned} \text{maximize:} \quad & \lambda \\ \text{subject to:} \quad & M_{i,j}(g) = \langle \delta_{i,j} \otimes \delta_g, P \rangle, \\ & P \succeq 0. \end{aligned}$$

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Corollary

The first cohomology of $\mathrm{SL}_3(\mathbb{Z})$ vanishes, and the second is reduced.

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- Expanders generalize to higher dimensions (Lubotzky)
- $SL_3(\mathbb{Z})$: spectral gap $\Rightarrow$ "CW-expanders"

Thank you for attention!